SS 2016 Dr. Milena Pabiniak
Marc Kegel

Differential Geometry

Homework 9

Mandatory Exercise 1. (10 points)
Consider the usual local coordinates (θ, φ) in $S^2 \subset \mathbb{R}^3$ defined by the parametrization $\phi\colon (0,\pi) \times (0, 2\pi) \to \mathbb{R}^3$ given by

$$\phi(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

(a) Using these coordinates, determine the expression of the Riemannian metric induced on S^2 by the Euclidean metric of $\mathbb{R}^3$.

(b) Compute the Christoffel symbols for the Levi-Civita connection in these coordinates.

(c) Show that the equator is the image of a geodesic.

(d) Show that any rotation about an axis through the origin in $\mathbb{R}^3$ induces an isometry of S^2.

(e) Show that the images of geodesics of S^2 are great circles.

(f) Find a **geodesic triangle** (i.e. a triangle whose sides are images of geodesics) whose internal angles add up to $\frac{3\pi}{2}$.

(g) Let $c\colon \mathbb{R} \to S^2$ be given by $c(t) = (\sin\theta_0 \cos t, \sin\theta_0 \sin t, \cos\theta_0)$, where $\theta_0 \in (0, \frac{\pi}{2})$ (therefore c is not a geodesic). Let V be a vector field parallel along c such that $V(0) = \partial_\theta$ (∂_θ is well defined at $(\sin\theta_0, 0, \cos\theta_0)$ by continuity). Compare the angle by which V is rotated when it returns to the initial point.

(h) Use this result to prove that no open set $U \subset S^2$ is isometric to an open set $W \subset \mathbb{R}^2$ with Euclidean metric.

(i) Given a geodesic $c\colon \mathbb{R} \to \mathbb{R}^2$ of $\mathbb{R}^2$ with Euclidean metric and a point $p \notin c(\mathbb{R})$, there exists a unique geodesic $\tilde{c}\colon \mathbb{R} \to \mathbb{R}^2$ (up to reparametrization) such that $p \in \tilde{c}(\mathbb{R})$ and $c(\mathbb{R}) \cap \tilde{c}(\mathbb{R}) = \emptyset$ (**parallel postulate**). Is this true in S^2?

Mandatory Exercise 2. (10 points)
Recall that identifying each point in $H = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ with the invertible affine map $h\colon \mathbb{R} \to \mathbb{R}$ given by $h(t) = yt + x$ induces a Lie group structure on H.

(a) Show that the left-invariant metric induced by the Euclidean inner product $dx \otimes dx + dy \otimes dy$ in $\mathfrak{h} = T_{(0,1)}H$ is

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy).$$

H endowed with this metric is called the **hyperbolic plane**.

(b) Compute the Christoffel symbols of the Levi-Civita connection in the coordinates (x, y).

(c) Show that the curves $\alpha, \beta\colon \mathbb{R} \to H$ given in these coordinates by

$$\begin{aligned} \alpha(t) =& (0, e^t) \\ \beta(t) =& \big(\tanh t, \frac{1}{\cosh t}\big) \end{aligned}$$

are geodesics. What are the sets $\alpha(\mathbb{R})$ and $\beta(\mathbb{R})$?

(d) Determine all images of geodesics.

(e) Show that, given two points $p, q \in H$, there exists a unique geodesic through them (up to reparametrization).

(f) Give examples of connected Riemannian manifolds containing two points through which there are (i) infinitely many geodesics (up to reparemetrization); (ii) no geodesics.

(g) Show that no open set $U \subset H$ is isometric to an open set $V \subset \mathbb{R}^2$ with the Euclidean metric.

(h) Does the parallel postulate hold in the hyperbolic plane?

Suggested Exercise 1. (0 points)
Show that in Euclidean space, the parallel transport of a vector between two points does not depend on the curve joining the two points. Show that this fact may not be true on an arbitrary Riemannian manifold.

Suggested Exercise 2. (0 points)
Let M be a Riemannian manifold. Consider the mapping

$$P = P_{c,t_0,t}\colon T_{c(t_0)}M \to T_{c(t)}M$$

defined by: $P_{c,t_0,t}(v)$, $v \in T_{c(t_0)}M$, is the vector obtained by parallel transporting the vector v along the curve c. Show that P is an isometry and that, if M is oriented, P preserves the orientation.

Suggested Exercise 3. (0 points)
Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, c an arbitrary parallel of a latitude on S^2 and V_0 a tangent vector to S^2 at a point of c. Describe geometrically the parallel transport of V_0 along c.
Hint: Consider the cone C tangent to S^2 along c and show that the parallel transport of V_0 along c is the same, whether taken relative to S^2 or to C.

Hand in: Monday 20th June
in the exercise session
in Seminar room 2, MI